ON STABILIZATION OF A NONHOLONOMIC SYSTEM

(O STABILIZATSII NEGOLONOMNOI SISTEMY)

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G.S.SHELEMENT'EV
(Sverdlovsk)

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In this paper we consider the problem of stabilization of nonholonomic systems in the neighborhood of the set of unstable positions of equilibrium, and construct a stabilizing control analytical in coordinates and velocities.

1. Let us consider a controlled mechanical system whose position is defined in terms of generalized coordinates $q_i(t)$ $(i = 1,\ldots, n+l)$ with l holonomic constraints. We shall assume that the constraints are linear and steady, and can therefore be represented as a system of l nonintegrable differential equations

$$\sum_{i=1}^{n+l} \varkappa_{ki}(q_1, \ldots, q_{n+l})\, q_i^{\cdot} = 0 \qquad (k = 1, \ldots, l) \tag{1.1}$$

where $\varkappa_{ki}(q)$ are functions of generalized coordinates q_i only. We shall assume that the forces acting on the system have a force function. Then, motion of the system under consideration can be described in terms of Lagrange equations with undetermined multipliers [1]

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = -\frac{\partial \Pi}{\partial q_i} + b_i u + \sum_{k=1}^{l} \lambda_k \varkappa_{ki} \qquad (i = 1, \ldots, n+l) \tag{1.2}$$

Here $T(q^{\cdot})$ and $\Pi(q)$ are the kinetic and potential energy of the system, u is a scalar defining the magnitude of the control, $b_i(q)$ are functions defining the direction of the control u in the space $\{q_i\}$, functions T, Π and b_i are given and λ_k are the undetermined multipliers. We shall consider here the case $l = 1$, but the arguments employed can be extended to more general cases without any fundamental difficulties. In absence of control $(u \equiv 0)$, the position of equilibrium of the system is defined by the set of equations

$$\frac{\partial \Pi}{\partial q_i} - \lambda \varkappa_i = 0 \qquad (i = 1, \ldots, n+1) \tag{1.3}$$

As we know [2 and 3], from this it follows that the equilibrium positions of a nonholonomic system are not isolated points, but form a manifold (which is, in our case, a one-parameter manifold) which, afer the elimination of λ

from (1.3), can be described, for definiteness, by (*)

$$q_i^0 = q_i^0(q_{n+1}), \qquad q^{\cdot 0}_j = 0 \qquad (i = 1,\ldots,n;\ \ j = 1,\ldots,n+1) \tag{1.4}$$

We shall assume that the equilibrium position (1.4) of the system (1.2) is unstable for each value $q_{n+1} = q$ belonging to some interval $\alpha < q < \beta$. With this assumption we can formulate the problem of stabilization ([4], p.476) of the system (1.1) and (1.2), i.e. the problem of selecting such a controlling force $u(q_1, \ldots, q_{n+1}, q_1^{\cdot}, \ldots, q_{n+1}^{\cdot})$, under the action of which the position of equilibrium would become asymptotically stable. We know, that in the absence of control u, full asymptotic stability of each isolated position of equilibrium (1.4) of a nonholonomic system cannot be achieved [2 and 3]. An obvious inference suggests itself, that such stability cannot be achieved even under the action of control $u = u(q, q^{\cdot})$.

We shall elaborate this point by considering this situation from the point of view of the theory of stabilization, developed in [5].

Let us assume that the equilibrium position of (1.4) corresponding to the value $q_{n+1} = q^*$ is unperturbed and let us construct the equation of perturbed motion [4 and 6] of the system (1.1) and (1.2) in the neighborhood of this position $q_i^\circ = q_i^\circ(q^*)$ $(i = 1, \ldots, n+1)$. Changing to new coordinates $s_i = q_i - q_i^\circ(q^*)$ and eliminating the undetermined multiplier λ, we obtain the following equations of motion:

$$\frac{d}{dt}\frac{\partial T}{\partial s_i^{\cdot}} - \frac{\partial T}{\partial s_i} = -\frac{\partial \Pi}{\partial s_i} + b_i u + \left[\frac{d}{dt}\frac{\partial T}{\partial s_{n+1}^{\cdot}} - \frac{\partial T}{\partial s_{n+1}} + \frac{\partial \Pi}{\partial s_{n+1}} - b_{n+1} u\right]\omega_i \tag{1.5}$$

$$(i = 1, \ldots, n)$$

$$s_{n+1}^{\cdot} = \sum_{i=1}^{n} \omega_i s_i^{\cdot} \tag{1.6}$$

When $u = 0$, system (1.5) is in equilibrium $s_j = 0$. Let us introduce a new variable

$$\xi = s_{n+1} - \sum_{i=1}^{n} \omega_i^\circ s_i$$

Here ω_i° are the values of the function $\omega_i(s_1, \ldots, s_{n+1})$ in (1.6), when the coordinates s_i are equal to zero. Then, assuming that the kinetic and potential energy of the system is

$$2T = \sum_{i,j=1}^{n+1} a_{ij}(s_1, \ldots, s_{n+1}) s_i^{\cdot} s_j^{\cdot}, \qquad 2\Pi = \sum_{i,j=1}^{n+1} b_{ij}(s_1, \ldots, s_{n+1}) s_i s_j$$

we obtain the following system of equations of motion:

$$\sum_{j=1}^{n} \alpha_{ij} s_j^{\cdot\cdot} = \sum_{j=1}^{n} b_{ij} s_j + \beta_i u + \gamma_i \xi + \varphi_i(\xi, s, s^{\cdot}, u) \qquad (i = 1, \ldots, n) \tag{1.7}$$

$$\xi^{\cdot} = \sum_{k+r=1}^{\infty} \sum_{i,j=1}^{n} \omega_{ij}^{(kr)} \xi^k s_i^r s_j^{\cdot} \tag{1.8}$$

It should be noted that the expansion of the right-hand side of (1.8) begins with the terms of at least second order in coordinates and velocities.

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*) Following the example of [2], here and in the following we disregard the special cases.

Solving (1.7) with respect to higher derivatives and substituting $s_i = x_{2i-1}$, $s_i^{\cdot} = x_{2i}$ $(i = 1, \ldots, n)$, we obtain (1.7) in the form

$$x_{2i-1}^{\cdot} = x_{2i}, \quad x_{2i}^{\cdot} = \sum_{j=1}^{n} a_{i,\,2j-1} x_{2j-1} + b_i u + c_i \xi + \varphi_i(\xi, x, u) \tag{1.9}$$

Suppose now that the subsystem

$$x^{\cdot} = A(q^*)\,x + b(q^*)\,u \qquad (x = \{x_1, \ldots, x_{2n}\}) \tag{1.10}$$

obtained from (1.9) by putting $\xi = 0$ and neglecting φ_i, satisfies the conditions of stabilization [5 and 7]. Then, its unperturbed motion can be made asymptotically stable by means of the control

$$u_*(x) = \sum_{i=1}^{2n} p_i x_i \tag{1.11}$$

We shall seek, for the complete system (1.8) and (1.9), a stabilizing action u in the form

$$u(x, \xi) = u_*(x) + u(\xi, |\xi|) \tag{1.12}$$

admitting also the functions which are not analytic

$$u(\xi, |\xi|) = \sum_{i=1}^{\infty} \alpha_i \xi^i + \sum_{i=1}^{\infty} \beta_i |\xi|^i \tag{1.13}$$

This widens, in some cases, the possibilities of stabilizing the system [5]. In accordance with the theory of critical cases [6 and 8], we shall introduce the following Liapunov transformation

$$\xi = \xi, \qquad x_i = z_i + w_i(\xi) \tag{1.14}$$

where z_i is a new variable and the functions $w_i(\xi)$ satisfy, in the neighborhood of the coordinate origin $x = 0$, $\xi = 0$, Equations

$$x_{2i} = 0, \qquad \sum_{j=1}^{2n} a_{i,\,2j-1} x_{2j-1} + c_i \xi + b_i (u_*(x) + u(\xi, |\xi|)) + \varphi_i(\xi, |\xi|, x) = 0 \tag{1.15}$$

Using (1.14) we obtain the system (1.8) and (1.9) in the standard form

$$z_i^{\cdot} = \sum_{j=1}^{2n} d_{ij} z_j + \Phi_i(\xi, z_1, \ldots, z_{2n}), \qquad \xi^{\cdot} = \Phi(\xi, z_1, \ldots, z_{2n})$$

discovering at the same time, that the function $\Phi^\circ(\xi, z_1, \ldots, z_{2n}) = \Phi(\xi, 0, \ldots, 0) \equiv 0$ for any values of α_i, and β_i, in (1.7). This represents the special case, when the control (1.12) does not make the system (1.8) and (1.9) asymptotically stable. Nevertheless, as it always happens in such cases, when $u = u_*(x)$, then by (1.11), the unperturbed motion $x = 0$, $\xi = 0$ will be stable in the Liapunov sense, and every perturbed motion sufficiently close to it, will approach asymptotically some stationary motion $\xi = q$, $x_i = 0$ near the point $\xi = 0$, $x = 0$.

It follows, that in our case it is expedient to formulate the question of constructing the control $u = u(q, q^{\cdot})$ which stabilizes asymptotically not separate points $q = q^*$, $q_i = q_i^\circ(q^*)$, but the whole manifold of equilibrium positions $q_i^\circ(q)$ simultaneously. It is the construction of such control, that is the aim of this paper.

2. We shall adopt the following definition. Definition 2.1. The manifold of equilibrium positions

$$q^0{}_i = q_i^\circ(q) \qquad (\mu \leqslant q \leqslant \nu, \;\; \alpha < \mu < \nu < \beta; \;\; i = 1, \ldots, n) \tag{2.1}$$

of a nonholonomic system (1.1) and (1.2) shall be called asymptotically stable, if every position is stable in the Liapunov sense and, if for each perturbed motion close to any of these equilibria $q_1{}^{\circ}(q^*)$, the condition

$$\lim q_i = q_i{}^{\circ}(q^* + \varepsilon) \quad \text{for } t \to \infty, \qquad \lim q = q^* + \varepsilon \quad \text{for } t \to \infty$$

is fulfilled.

Here ε is an arbitrary small quantity provided the initial perturbations $q_1 - q_1{}^{\circ}(q^*)$ and $q - q^*$ are small. Let us now consider the following problem.

P r o b l e m 2.1 . To find the control $u = u(q_1, q_1^{\cdot}, q, q^{\cdot})$ such, that the manifold of equilibrium points $q_1 = q_1{}^{\circ}(q)$ becomes asymptotically stable in the sense of the definition 2.1.

We have established before, that, provided that the system (1.10) can be stabilized, for each fixed value $q = q^*$ a control $u = u_*(x) = u(x, q^*)$ exists which, under small deviations of q from q^* and x from $x(q^*)$, brings the system into the equilibrium position in the neighborhood of the point $q = q^*$, $x = x(q^*)$. We find therefore, that in order to solve the problem 2.1, it is sufficient to show that the control u can be constructed in the form of sufficiently well defined function of parameter q

$$u = \sum_{i=1}^{2n} p_i(q)\, x_i \tag{2.2}$$

Ideed we find that the control (2.2) can be constructed in the form of a function analytic in q , for every q belonging to the segment $[\gamma, \vartheta]$, $(\alpha < \gamma < \mu < \nu < \vartheta < \beta)$ To show this, we shall utilize the method of Liapunov functions and its connection with the method of dynamic programing [9]. Let the system (1.10) satisfy the conditions for stabilization [5 and 7] for each fixed q from the segment $[\gamma, \vartheta]$. We shall seek the functions $V(x, q)$ and $u(x, q)$ satisfying the criteria of an optimum [4] of motions of (1.10) in the problem on the minimum of the following integral:

$$\int_0^{\infty} \Big(\sum_{i=1}^{2n} x_i^2 + u^2\Big) dt \tag{2.3}$$

The fact that these functions are analytic in q , is important. Indeed, we have the following Lemma.

L e m m a 2.1 . Let the problem on optimal stabilization of (1.10) and (2.3) possess a solution when $q = q^*$, and let us denote by

$$V(x, q^*) = \sum_{i,j=1}^{2n} \alpha_{ij}(q^*)\, x_i x_j$$

the optimal Liapunov function satisfying all the criteria of stabilization. Then, such $\delta > 0$ can be found, that the analogous problem on the optimal stabilization of the system

$$\dot{x} = A(q)\, x + b(q) u \tag{2.4}$$

can be solved for all q lying in the δ-neighborhood of the point q^*, and for these q we can construct optimal Liapunov functions

$$V(x, q) = \sum_{i,j=1}^{2n} \alpha_{ij}(q) x_i x_j \tag{2.5}$$

the coefficients $\alpha_{ij}(q)$ of which can be expanded into power series

$$\alpha_{ij}(q) = \sum_{k=0}^{\infty} \alpha_{ij}^{(k)} (q - q^*)^k \tag{2.6}$$

converging in the δ-neighborhood of q^*.

P r o o f . Let us expand the functions $a_{ij}(q)$ and $b_i(q)$ into a series in the neighborhood of the point $q = q^*$ Putting $q - q^* = y$, we obtain

$$a_{ij}(q) = \sum_{k=0}^{\infty} a_{ij}^{(k)} y^k, \qquad b_i(q) = \sum_{k=0}^{\infty} b_i^{(k)} y^k \tag{2.7}$$

Insertion of (2.7) into (2.4) results in

$$\frac{dx_i}{dt} = \sum_{j=1}^{2n} a_{ij}^{(0)} x_j + b_i^{(0)} u + \sum_{k=1}^{\infty} \sum_{j=1}^{2n} y^k a_{ij}^{(k)} x_j + \sum_{k=1}^{\infty} y^k b_i^{(k)} u \tag{2.8}$$

When $y = 0$ (i.e. when $q = q^*$), (2.8) becomes (2.4), and the problem of optimal stabilization has a solution, also the optimal Liapunov function $V_0(x, q^*)$ exists. We shall show that the complete system (2.8), also has an optimal Liapunov function

$$V(x, y) = \sum_{k=0}^{\infty} \sum_{i,j=1}^{2n} y^k \alpha_{ij}^{(k)} x_i x_j = \sum_{k=0}^{\infty} y^k V_k(x) \tag{2.9}$$

provided y is sufficiently small. The Liapunov-Bellman equations

$$\min_u \Bigg[\sum_{i,j=1}^{2n} a_{ij}^{(0)} x_j \frac{\partial V}{\partial x_i} + u \sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V}{\partial x_i} + \sum_{i=1}^{2n} x_i^2 + u^2 + \\ + \sum_{k=1}^{\infty} \sum_{i,j=1}^{2n} y^k a_{ij}^{(k)} x_j \frac{\partial V}{\partial x_i} + u \sum_{k=1}^{\infty} \sum_{i=1}^{2n} y^k b_i^{(k)} \frac{\partial V}{\partial x_i} \Bigg] = 0 \tag{2.10}$$

$$u = -\frac{1}{2} \sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V}{\partial x_i} - \frac{1}{2} \sum_{k=1}^{\infty} \sum_{i=1}^{2n} y^k b_i^{(k)} \frac{\partial V}{\partial x_i} \tag{2.11}$$

yield the following partial differential equations

$$\sum_{k=0}^{\infty} \sum_{i,j=1}^{2n} y^k a_{ij}^{(k)} x_j \frac{\partial V}{\partial x_i} + \sum_{i=1}^{2n} x_i^2 - \frac{1}{4} \Big(\sum_{k=0}^{\infty} \sum_{i=1}^{2n} y^k b_i^{(k)} \frac{\partial V}{\partial x_i} \Big)^2 = 0 \tag{2.12}$$

from which Liapunov function $V(x)$ can be determined.

Solution of (2.12) will be sought in form of the series (2.9)

$$V(x, q) = \sum_{k=0}^{\infty} \sum_{i,j=1}^{2n} y^k \alpha_{ij}^{(k)} x_i x_j \,.$$

When $y = 0$, the above equation has, by the condition of our Lemma, a solution $V = V_0(x, q^*)$, which can be found from

$$\sum_{i,\, j=1}^{2n} a_{ij}^{(0)} x_j \frac{\partial V_0}{\partial x_i} + \sum_{i=1}^{2n} x_i^2 - \frac{1}{4}\left(\sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V_0}{\partial x_i}\right)^2 = 0$$

Optimal control $u^{(0)}$ is equal to

$$u^{(0)}(x,\, q^*) = -\frac{1}{2}\sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V_0}{\partial x_i}$$

and the system

$$\frac{dx_i}{dt} = \sum_{j=1}^{2n} a_{ij}^{(0)} x_j - \frac{1}{2} b_i^{(0)} \sum_{j=1}^{2n} b_j^{(0)} \frac{\partial V_0}{\partial x_j} \qquad (i = 1, \ldots, 2n) \tag{2.13}$$

is asymptotically stable.

Inserting (2.9) into (2.11) and equating the coefficients of y to zero, we obtain the following equations for $V_k(x)$:

$$\sum_{i,\, j=1}^{2n} a_{ij}^{(0)} x_j \frac{\partial V_0}{\partial x_i} + \sum_{i=1}^{2n} x_i^2 - \frac{1}{4}\left(\sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V_0}{\partial x_i}\right)^2 = 0$$

$$\sum_{s=0}^{k} \sum_{i,\, j=1}^{2n} a_{ij}^{(s)} x_j \frac{\partial V_{k-s}}{\partial x_i} - \frac{1}{4} \sum_{p,\, q\geqslant 0}^{p+q=k} \left(\sum_{s=0}^{p} \sum_{i=1}^{2n} b_i^{(s)} \frac{\partial V_{p-s}}{\partial x_i}\right)\left(\sum_{m=0}^{q} \sum_{i=1}^{2n} b_i^{(m)} \frac{\partial V_{q-m}}{\partial x_i}\right) = 0$$

$$(k = 1, 2, \ldots .)$$

Since the derivative of $V_k(x)$ with respect to time has, by virtue of (2.13) the form

$$\left[\frac{dV_k}{dt}\right]_{(2.13)} = \sum_{i,\, j=1}^{2n} a_{ij}^{(0)} x_j \frac{\partial V_k}{\partial x_i} - \frac{1}{2}\left(\sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V_k}{\partial x_i}\right)\left(\sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V_0}{\partial x_i}\right) \tag{2.14}$$

we find the terms of the series (2.9) from

$$\left[\frac{dV_0}{dt}\right]_{(2.13)} = -\sum_{i=1}^{2n} x_i^2 - \frac{1}{4}\left(\sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V_0}{\partial x_i}\right)^2$$

$$\left[\frac{dV_k}{dt}\right]_{(2.13)} = -\sum_{s=1}^{k} \sum_{i,\, j=1}^{2n} a_{ij}^{(s)} x_j \frac{\partial V_{k-s}}{\partial x_i} - \frac{1}{2}\left(\sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V_0}{\partial x_i}\right)\left(\sum_{i=1}^{2n} b_i^{(0)} \frac{\partial V_k}{\partial x_i}\right) + \tag{2.15}$$

$$+\frac{1}{4} \sum_{p,\, q\geqslant 0}^{p+q=k} \left(\sum_{s=0}^{p} \sum_{i=1}^{2n} b_i^{(s)} \frac{\partial V_{p-s}}{\partial x_i}\right)\left(\sum_{m=0}^{q} \sum_{i=1}^{2n} b_i^{(m)} \frac{\partial V_{q-m}}{\partial x_i}\right)$$

As (2.13) is asymptotically stable, we can use the Liapunov theorem ([4], p.67) to prove the existence of a unique solution of (2.15) with $k \geq 0$ arbitrary. Hence, knowing $V_k(x)$ we can find $u_k(x)$ and consecutive terms of (2.9) of any order. We find during this procedure, that a unique solution to the problem exists at every step. Convergence of the series is proved by the method similar to that used in the analogous case in [10 and 11].

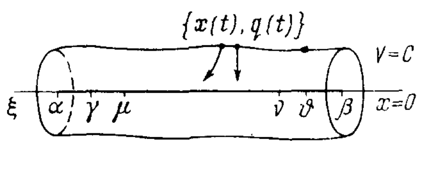

Fig. 1

With the series (2.9) constructed, function $u^{\circ}(x,\, y)$ can be found from (2.11). Consequently, the series for u° has the form

$$u^{\circ}(x, y) = \sum_{k=0}^{\infty} y^k u_k^{\circ}(x) \tag{2.16}$$

where

$$u_k^{\circ}(x) = -\frac{1}{2}\sum_{i=1}^{2n} b_i^{(k)} \frac{\partial V_k}{\partial x_i} \tag{2.17}$$

Since (2.9) converges, (2.16) and (2.17) also converge in u° .

Proof of asymptotic stability of the manifold $q_1 = q_1^{\circ}(q)$ in the sense of 2.1 follows that of the analyticity of the control $u(x, q)$ stabilizing the system (2.4) for every fixed q . To achieve this the usual [2 to 4, and 6] examples are used, and the proof is based on the constructed function $V(q_1, q_1^{\cdot}, q)$, the total differential dV/dt of which is, by virtue of equations of motion for $u = u(x, q)$, negative. Since the proof follows closely that of [2], we shall omit it, and mention instead some geometrical aspects of our phenomenon. Since V is analytic in q and x , surfaces $V(x, q) = C$ form, in the $\{x, q\}$ space, smooth tubes enclosing the line $x = 0$ (Fig.1).

Smoothness of these tubes, the condition that for a fixed $q = q^*$ we have

$$\left(\frac{dV}{dt}\right)_{q=q^*} = -\left(\sum_{i=1}^{2n} x_i^2 + u^2\right)$$

and the equality $\Phi(\xi, 0, \ldots, 0) = 0$, together imply that the motion $\{x(t), q(t)\}$ is directed along the tube, and its depth of penetration into the tube during the time dt , is of first order in $\|x(t)\|$. From this we infer that our previous statement was correct, and that the following theorem is true.

T h e o r e m 2.1 . If the system (1.10) satisfies the conditions for stabilization [4] for all fixed q belonging to the segment $[\gamma, \vartheta]$ then, for sufficiently small y , a control $u = u(q_1, q_1^{\cdot}, q)$ exists, which is a solution of the problem 2.1, and is in the form of (2.16) and (2.17).

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